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Annuity Choices for Pensioners

M. Zaki Khorasane*^{*}

Abstract[†]

We consider two ways for a retiree to obtain a pension from a retirement fund: through the purchase of a whole life annuity providing a level monetary income; and through the withdrawal of income from a fund invested in equities. Deterministic and stochastic models are used to assess the risks and benefits associated with each approach. In each case the projected cash flows are compared with those from a whole life annuity providing an income linked to price inflation. We conclude that, although each of the two options considered involves significant risks, each method may be attractive to certain groups of pensioners, particularly those with additional savings held outside the retirement fund.

Key words and phrases: *index-linked, inflation, equity portfolio, sinking fund, break-even duration*

1 Introduction

In the United Kingdom (U.K.) certain types of pension plans provide a lump sum benefit at retirement rather than monthly (or other periodic) payments.¹ The manner in which the lump sum is invested

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¹These include individual insured plans, defined contribution plans, and defined benefit plans in which the benefit formula is for a lump sum at retirement.

is restricted by legislation. A retirement fund must be established to provide income although a portion of the fund, usually not exceeding 25 percent, may be taken in cash. The safest option may be to buy a whole life annuity providing an income guaranteed to increase with an index of consumer price inflation. In the U.K. such a product is called an *index-linked annuity*.²

This paper considers the merits of two alternatives to the index-linked annuity that currently are available in the U.K.:

- A level whole life annuity providing a stable income, the most popular option; and
- An income withdrawal option in which no annuity is purchased, but the pensioner draws an income from the retirement fund.

We examine the implications of each by comparing the projected cash flows against those from an index-linked annuity.

1.1 Why Not Choose an Index-Linked Annuity?

Index-linked annuities are relatively unpopular with U.K. consumers because the income obtained is initially much lower than the income provided by a level annuity. The following table shows the initial income available from the most competitive U.K. insurer in July 1996 for each type of annuity.³

In July 1996 consumer price inflation was approximately 3 percent per annum in the U.K. If this rate of inflation were to continue, an index-linked annuity would provide a lower income for 13 years from age 60, 11 years from age 65, and 10 years from age 70. Pensioners may believe that even if they live a reasonable life span beyond these durations, they probably would receive a higher aggregate income from a level annuity, and we show that this opinion is justified.⁴

The purchase of a level annuity has two potential disadvantages:

- The pensioner may live well beyond his/her life expectancy;

²The index used is the Retail Prices Index. It is the most widely used measure of U.K. consumer price inflation and is published monthly by the government.

³Retiring members are normally given an open market option, i.e., the right to buy an annuity with the insurance company of their choice. As a consequence, most U.K. life annuity business is written by a relatively small number of insurers offering the most competitive rates.

⁴The apparently poor rates offered for index-linked annuities may be a consequence of mortality selection—lives in poor health are unlikely to opt for index-linked annuities. In analyzing the choice for any one individual, however, we assume the mortality of that individual will be the same whichever type of annuity is purchased.

Table 1
Annual Income to a Male
Pensioner From a £10,000 Fund

Retirement Age	Level Annuity	Index-Linked Annuity
60	£1,016	£683
65	£1,119	£797
70	£1,280	£951

Source: Annuity Direct (July 19, 1996), available from Ceefax, a teletext service of the British Broadcasting Corporation (BBC).

- Inflation may be higher than anticipated.

For these reason, purchasers of level annuities are sometimes advised to save a substantial part of the income they obtain in the early years to build a fund that can provide some protection against inflation and longevity. In a later section we examine the efficacy of such a policy by assuming that pensioners save (or dis-save) the difference between the income from their level annuity and the income that would have been obtained from an index-linked annuity. Even such a conservative policy does not immunize the pensioner against risk.

1.2 The Income Withdrawal Option

The income withdrawal option was introduced in the U.K. in 1995. This option allows the pensioner to defer the purchase of an annuity, instead the pensioner draws an income by utilizing assets in the fund. Certain restrictions apply, however: the income drawn must lie between 35 percent and 100 percent of the income that otherwise could have been obtained from a level annuity, and the pensioner cannot delay the purchase of a whole life annuity beyond age 75.

The income withdrawal option has two principal attractions:

- At the time of death, the capital remaining in the fund remains part of the pensioner's estate; and
- The fund can be invested in assets that may provide a higher return to the pensioner than available from a whole life annuity.

The first characteristic is sometimes described as *capital protection*. In a whole life annuity, the absence of a death payout allows a higher

level of income for the living. Thus, the income from a whole life annuity fund will exceed that from an income withdrawal fund if both are invested in the same assets and are subject to the same expenses. This difference increases as the pensioner ages because of the increasing mortality strain.

It follows that an income withdrawal fund must earn a higher return than a whole life annuity fund if the pensioner is to maintain an equivalent income. In practice, this means that an income withdrawal fund is likely to be invested wholly or partly in equities, as equities are expected to outperform the government bonds held by insurers' annuity funds. The market price volatility associated with equities, however, creates additional risks for the pensioner.

2 Level vs. Index-Linked Annuities

2.1 Assets of Whole Life Annuity Fund

The income received from a whole life annuity policy depends on the interest, mortality, and expenses assumed by the insurer for its annuity portfolio. The interest assumption depends on the assets held by the insurer. In the case of level annuities, the insurer normally holds a fixed interest bond portfolio roughly matching the mean term of its liabilities. In the case of index-linked annuities, the insurer normally holds index-linked bonds, providing interest and principal payments that increase in step with consumer price inflation. Both types of securities have been issued by the British government to finance its national debt, although the total market value of fixed interest bonds currently in issue is roughly eight times as great as that of index-linked bonds.⁵

The prospective return on a fixed interest bond is measured by its *gross redemption yield*.⁶ The prospective return offered by an index-linked bond is measured relative to future price inflation and is termed the *real gross redemption yield*. This real yield can be thought of as the interest rate at which the present value of future income and capital payments from the bond would equal its current market value if future inflation were zero. The interest rates used to price level and index-linked annuities closely follow the average yields available on fixed interest and index-linked bonds, respectively.

⁵Total market capitalizations were £217bn (\$337bn) for fixed interest bonds and £26bn (\$41bn) for index-linked bonds (*The Times of London*, July 29, 1996, daily business section).

⁶The *gross redemption yield* is the interest rate at which the present value of future income and capital payments equals the market value of the bond.

We now define the annual price inflation and other annual rates:

- j = Interest rate used to price level annuities;
- c = Expected rate of price inflation; and
- r = Real interest rate used to price index-linked annuities.

In a scenario of constant interest rates and constant price inflation, we would expect the total return on fixed interest and index-linked bonds to be identical, thus:

$$(1 + j) = (1 + r)(1 + c). \quad (1)$$

Throughout this section we assume that the difference between the nominal yield on fixed interest bonds and the real yield on index-linked bonds gives an unbiased estimate of future price inflation.

2.2 Deterministic Comparison

We now compare the projected income from both types of annuities assuming a constant force of price inflation. The following notation will be used:

- t = Time since the purchase of a level annuity, $t \geq 0$;
- x = The age of a pensioner at the moment of retirement;
- \bar{a}_x^r = Purchase price of an indexed-linked annuity of 1 per annum payable to a pensioner age x at retirement;

$$= \int_0^\infty (1 + c)^t (1 + j)^{-t} {}_t p_x dt = \int_0^\infty (1 + r)^{-t} {}_t p_x dt$$
- \bar{a}_x^j = Purchase price of a continuous level annuity of 1 per annum payable to a pensioner age x at retirement;

$$= \int_0^\infty (1 + j)^{-t} {}_t p_x dt$$
- $(AVS)_t$ = Accumulated value of sinking fund t years after the purchase of a level annuity.

We assume that, to maintain his/her standard of living, the purchaser of a level annuity actually spends (consumes) money at a rate

equal to the income generated from an index-linked annuity, the difference being saved in (or withdrawn from) a sinking fund. Thus if a retirement fund of one monetary unit is used to buy a level annuity providing a continuous income, the rate at which income is saved in (or withdrawn from) the sinking fund at time t is given by $S_t(j, c)$:

$$S_t(j, c) = \frac{1}{\bar{a}_x^j} - \frac{(1+c)^t}{\bar{a}_x^r}.$$

The present value of the sinking fund accumulation up to time t , discounted back to the purchase date at the nominal interest rate, j , is:

$$\begin{aligned} (\text{PVS})_t(j, c) &= \int_0^t S_\tau(j, c)(1+j)^{-\tau} d\tau \\ &= \frac{\bar{a}_{t|}^j}{\bar{a}_x^j} - \frac{\bar{a}_{t|}^r}{\bar{a}_x^r}. \end{aligned} \quad (2)$$

Initially the sinking fund grows because the money spent by the pensioner is less than the income from the level annuity. As the payment from the index-linked annuity increases continuously at a constant rate, it will eventually exceed the level annuity payments.⁷ Thus if the pensioner lives for a sufficiently long time, the sinking fund will become zero and the pensioner will have to withdraw money from the sinking fund.

Let N , called the *break-even duration*, denote the first time the sinking fund falls to zero. Thus the level annuity provides a more valuable overall income for a pensioner who dies before N , and the index-linked annuity provides a more valuable overall income for a pensioner who survives beyond N . As $(\text{PVS})_N(j, c) = 0$, equation (2) yields

$$\frac{\bar{a}_{N|}^j}{\bar{a}_{N|}^r} = \frac{\bar{a}_x^r}{\bar{a}_x^j}. \quad (3)$$

To price these annuities, let us assume $j = 7$ percent net of all expenses; $r = 3$ percent net of all expenses; $c = 1.07/1.03 - 1 = 3.88$ percent; and mortality following the male PA(90) life table.⁸ Using standard numerical methods, equation (3) can be solved for N . The results

⁷In fact the index-linked annuity payment ultimately is unbounded.

⁸The PA(90) life table is based on the mortality experience of pensioners in employer-sponsored plans administered by U.K. insurance companies. The table is published by the Institute and Faculty of Actuaries.

are shown in Table 2. In addition Table 2 shows the life expectancy at retirement (\dot{e}_x), N , and the probability of surviving to age $x + N$, for different values of x . Notice that the sinking fund is exhausted one to three years after the pensioner's life expectancy at retirement. The fifth column of Table 2 shows that a retiree has less than a 50 percent chance of surviving to the age $x + N$, and this probability decreases with the age at retirement. A pensioner probably will obtain more value from a level

Table 2
Break-Even Durations, N

x	\dot{e}_x	N	${}_Np_x$	$(1 + c)^{-N}$
50	26.1	27.8	0.465	0.347
55	21.8	24.1	0.433	0.399
60	18.0	20.7	0.401	0.454
65	14.6	17.5	0.368	0.514
70	11.6	14.5	0.342	0.576
75	9.0	11.8	0.303	0.638

annuity than from an index-linked annuity.⁹ The risk of a dramatic fall in income on surviving beyond the break-even duration is significant at all ages. If a pensioner who bought a level annuity at 65 survived to the break-even duration, his or her rate of income at the break-even duration would be reduced by a factor of $(1 + c)^{-N} = 0.514$.

2.3 Effect of Uncertain Inflation

We now use a simple model to investigate the effect of uncertainty in the rate of inflation on our previous results. Let us again assume a constant inflation rate, but one that is different from the expected rate, i.e., let b denote the *actual* rate of price inflation. Using the actual inflation rate, equation (2) is modified as follows:

$$S_t(j, b) = \frac{1}{\bar{a}_x^j} - \frac{(1 + b)^t}{\bar{a}_x^r}$$

$$(\text{PVS})_t(j, b) = \int_0^t S_\tau(j, b)(1 + j)^{-\tau} d\tau$$

⁹By which we mean that the pensioner could reproduce the income from an index-linked annuity and still have a positive sinking fund at death.

$$= \frac{\bar{a}_{t|}^j}{\bar{a}_x^j} - \frac{\bar{a}_{t|}^{r'}}{\bar{a}_x^{r'}} \quad (4)$$

where $r' = (1 + j)/(1 + b) - 1$ is the actual real rate of interest. Again we can solve the equation $(PVS)_N(j, b) = 0$ to obtain the break-even duration.

More realistically, b will not be known at $t = 0$ because the actual rate of inflation is a random variable. Let B be the random variable denoting the actual rate of inflation, and assume B has a known discrete¹⁰ distribution $f_i = \Pr[B = b_i]$, for $i = \dots, -2, -1, 0, 1, 2, \dots$. Given that $B = b$, let $N(b)$ be the resulting break-even duration calculated according to the equation

$$(PVS)_{N(b)}(j, b) = 0. \quad (5)$$

There are two quantities of interest to us:

$$\begin{aligned} E[N(B)] &= \text{The expected break-even duration;} \\ &= \sum_{-\infty}^{\infty} N(b_i) f_i \quad \text{and} \end{aligned} \quad (6)$$

$$\begin{aligned} \pi_x &= \text{Probability of a pensioner who retires at age } x \\ &\quad \text{survives beyond exhaustion of the sinking fund;} \\ &= \sum_{-\infty}^{\infty} N(b_i) p_x f_i. \end{aligned} \quad (7)$$

As an example, consider the following case: $x = 65$, $j = 7$ percent, $c = 3.88$ percent, $B = c + 1$ percent with probability 0.25, $B = c$ with probability 0.50, and $B = c - 1$ percent with probability 0.25. Using each of the three possible values of B , equation (5) yields the information in Table 3.

$$\begin{aligned} E[N(B)] &= 26.0 \times 0.25 + 17.5 \times 0.50 + 13.3 \times 0.25 \\ &= 18.575 \\ \pi_{65} &= 26.0 p_{65} \times 0.25 + 17.5 p_{65} \times 0.50 + 13.3 p_{65} \times 0.25 \\ &= 0.093 \times 0.25 + 0.368 \times 0.50 + 0.543 \times 0.25 \\ &= 0.343. \end{aligned}$$

This probability of 0.343 is lower than the corresponding figure in Table 2, for two reasons:

¹⁰In practice, B has a continuous distribution.

Table 3
Exhausted Sinking Fund at N

b	$\Pr[B = b]$	$N(b)$	$N(b)p_{65}$
2.88%	0.25	26.0	0.093
3.88%	0.50	17.5	0.368
4.88%	0.25	13.3	0.543

- The break-even duration increases in the low inflation scenario by more than it falls in the high inflation scenario;
- Mortality increases with age, so the probability of surviving to higher ages falls rapidly.

These two effects combine to make the reduction in survival probability for the low inflation scenario greater than the increase in survival probability for the high inflation scenario. It follows that uncertainty in the rate of inflation may reduce the risk associated with a level annuity.

2.4 Stochastic Comparison

As the annual rate of inflation forms a sequence of random variables, it is difficult to quantify the risk of receiving inadequate income associated with a level annuity. Thus simulations are used to aid us in quantifying this risk.

In Sections 2.2 and 2.3 it was convenient to use a continuous-time model because most annuities bought in the U.K. provide a monthly income, making a continuous time model a reasonably good approximation of reality. In this section, however, we switch to a discrete time model for mathematical convenience. In addition, we assume that the annuity payments are made at the start of each year.

2.4.1 Inflation History of the United Kingdom

Parsons (1990) has examined the history of U.K. consumer price inflation since 1810 using the most representative index available in each era. From these data, he concludes that persistent positive inflation rates, as now exist in the U.K. and other industrialized economies, have only been observable since World War II. He therefore does not reject the possibility that inflation eventually may revert to its earlier pattern of behavior, in which the price level is as likely to fall as to rise in any

year and the average rate of inflation measured over long periods is small.

Another feature of the data is the existence of inflation shocks which appear to arise from the economic consequences of major historical events. Brief periods of high inflation occurred during all of the following crises:

- World War I;
- World War II;
- The rise in oil prices following the 1973 Arab/Israeli war; and
- The rise in oil prices at the start of Iran/Iraq war in 1980.

Currently, the rate of inflation is 3 percent per annum in the U.K., an historically low figure for the post-war era.

2.4.2 Scenarios for Future Inflation

For a pensioner retiring in current conditions there are three possible scenarios:

Scenario 1: The rate of inflation will continue to fall, and the economy will revert to long-term price stability;

Scenario 2: The economy will experience an inflation shock;

Scenario 3: Over the years, the rate of inflation will vary randomly within a few percentage points around a mean value close to the current rate.

The implications of the first two scenarios for the choice of annuity are clear. Under Scenario 1 fixed interest bonds would offer excellent real returns, and a level annuity would offer considerable extra value when compared with an index-linked annuity. Under Scenario 2 an index-linked annuity is the better choice to protect the real value of the pensioner's income. Under Scenario 3, however, the choice of annuity is unclear. Fortunately, this is the easiest scenario to model stochastically and is, perhaps, the most likely of the three scenarios to occur in the Britain.¹¹

¹¹In recent years a low and stable rate of inflation has been a stated objective of macroeconomic policy in Britain and other industrialized economies. The British government currently maintains a target range for inflation of 1 percent to 4 percent per annum.

2.4.3 Stochastic Model for Price Inflation

Stochastic models for the rate of inflation have been suggested by several authors including Wilkie (1986) and Clarkson (1991). Wilkie (1986) uses a first order autoregressive time series model¹² for price inflation for $t = 0, 1, 2, \dots$, as follows:

$$\ln(1 + b_{t+1}) = \mu_b + k_b[\ln(1 + b_t) - \mu_b] + \sigma_b Z_t \sqrt{1 - k_b^2} \quad (8)$$

where b_t is the annual rate of inflation over year t , i.e., over the interval $[t - 1, t)$ and the Z_t 's are a sequence of independent, identically distributed standard normal random variables (i.e., with mean zero and unit variance). In addition, μ_b and σ_b are the mean and standard deviation of $\ln(1 + b_t)$ respectively, and k_b is the correlation between $\ln(1 + b_t)$ in adjacent years.

The Wilkie model assumes that inflation varies around a long-term mean, in accordance with Scenario 3. It has been criticized by Clarkson (1991) for failing to allow for inflation shocks and other nonlinear effects. The difficulty with incorporating shock terms is that past inflation shocks arose from disparate and unique historical events, so they offer little help in modeling the future shocks.

Huber (1995) has also pointed out that the data used by Wilkie to parameterize his model included inflation shocks, which is inappropriate given that the model assumes a constant mean and variance for the rate of inflation. As the assumption of stationarity is only valid for periods of moderate inflation, the parameters should be estimated from periods that exclude inflation shocks.

Wilkie's approach is used to model inflation under Scenario 3, but the parameters are estimated from U.K. inflation rates since 1982 (excluding the inflation shock of 1980/81). The parameters estimates are $\mu_b = 0.047$, $\sigma_b = 0.019$ and $k_b = 0.58$. We also require the expected inflation rate to be consistent with the bases used for pricing level and index-linked annuities. Thus, we ignore the estimate from past data and assume, instead, that $\mu_b = \ln(1 + c)$. To summarize, the parameters used are

$$\mu_b = \ln(1 + c), \quad \sigma_b = 0.019 \quad \text{and} \quad k_b = 0.58. \quad (9)$$

It is possible that the variable c , as calculated from equation (1), may not give a realistic estimate of future inflation if it is derived from actual annuity rates available in the market, which are influenced by factors such as mortality selection, expenses, and competition. If so, a different

¹²For more on autoregressive time series, see, for example, Box and Jenkins (1976).

estimate could be used without affecting the method of comparison described in the next section.

2.4.4 Comparing Level and Index-Linked Annuities

For the deterministic comparison, we assume the pensioner buys a level annuity but only spends the income that would have been provided by an index-linked annuity, the difference being saved in (or withdrawn from) a sinking fund.

To simplify our simulations, we again assume the pensioner receives an income payable annually in advance. It follows that $(AVS)_t$, the sinking fund per unit of retirement fund just before the annuity payment at time t , must satisfy the following recurrence formula: $(AVS)_0 = 0$ and for $t = 0, 1, 2, \dots$

$$(AVS)_{t+1} = \left((AVS)_t + \frac{1}{\ddot{a}_x^j} - \frac{\beta_t}{\ddot{a}_x^r} \right) (1 + b_{t+1})(1 + r_{t+1}) \quad (10)$$

where r_t is the actual annual real interest earned on the sinking fund over the period $[t-1, t)$, b_t is the random actual annual rate of inflation over the period $[t-1, t)$, and

$$\beta_t = \prod_{\tau=1}^t (1 + b_\tau) \quad (11)$$

with $\beta_0 = 1$. The present value of the sinking fund, discounted using the actual interest rate earned on the sinking fund assets, is given by

$$(PVS)_t = \frac{(AVS)_t}{\beta_t R_t} \quad t = 0, 1, 2, \dots \quad (12)$$

where $R_0 = 1$ and

$$R_t = \prod_{\tau=1}^t (1 + r_\tau) \quad t = 1, 2, \dots \quad (13)$$

At this point, the natural question to ask is what rate of interest should the sinking fund earn? The interest earned depends on the investment vehicles used. Given the importance of the sinking fund to the pensioner, it would be prudent to place sinking fund savings in a relatively secure, interest-bearing deposit account—preferably an account where the interest earned is linked to the rate of inflation, as the purpose of the sinking fund is to provide long-term protection against

inflation. In the U.K. appropriate instruments are Index-Linked National Savings Certificates, which provide a fixed real rate of interest over five year periods. The real interest rate offered for the seventh issue of these certificates in 1993 was 3 percent per annum, close to the average real yield on index-linked bonds.

We therefore assume that the sinking fund assets achieve a fixed real yield equivalent to that assumed in the pricing basis for index-linked annuities, i.e., that $r_t = r$ for $t = 1, 2, \dots$

2.4.5 Results of Simulations

For these simulations we assume that the sinking fund can go into debt when its assets have been exhausted. Let $P_x(t, h)$ denote the probability that the present value of the sinking fund *debt* will exceed h at the end of t years, i.e.,

$$P_x(t, h) = \Pr[(PVS)_t < -h].$$

But this is also the probability that a sinking fund with initial assets at retirement equal to h would be exhausted after t years. Thus, $P_x(t, h)$ is the probability that, even with additional savings of h at retirement, the pensioner will not be able to maintain an inflation-linked income for t years.

Next let $P_x(h)$ be the probability that the sinking fund debt will exceed h at the end of the year of death. Assuming that the year of death does not depend on the rate of inflation, we can write:

$$P_x(h) = \sum_{t=0}^{\infty} P_x(t+1, h) {}_t p_x q_{x+t}. \quad (14)$$

To compute $P_x(h)$, we proceed as follows:

- Step 1:** Perform 1000 simulations for realizations of $(AVS)_t$ for $t = 1, 2, 3, \dots, 100 - x$ for selected retirement ages (x). These simulations are based on $j = 7$ percent and $r = 3$ percent and using the stochastic inflation model described in equations (8) and (9).
- Step 2:** For each x and t , use the simulated values of to construct an empirical distribution function for $(AVS)_t$.
- Step 3:** The empirical distribution functions are used to get a matrix of values of $P_x(t, h)$ for $h = 0.0, 0.1, 0.2, 0.3$ and $t = 1, 2, \dots, 100 - x$.

Table 4
Simulated Values of $P_x(h)$

x	h			
	0.00	0.10	0.20	0.30
50	0.38	0.17	0.05	0.01
55	0.33	0.15	0.05	0.01
60	0.31	0.14	0.05	0.02
65	0.29	0.14	0.05	0.02
70	0.24	0.11	0.05	0.02
75	0.22	0.10	0.05	0.02

Table 4 shows the values of $P_x(h)$ estimated from our simulations. Notice that the probabilities when $h = 0$ are lower than those obtained for the deterministic comparison, as we might expect from our earlier results in Section 2.3. The other columns show the probability of exhausting the sinking fund when it contains assets at retirement equal to h multiplied by the pensioner's retirement fund. An interesting feature of these simulations is that initial sinking fund assets equal to 20 percent of the pensioner's retirement fund would reduce the probability of exhaustion to 5 percent for all the retirement ages examined. Thus, for pensioners contemplating the purchase of a level annuity, additional liquid assets of roughly 20 percent of the retirement fund may insure reasonable protection against inflation in retirement, if one ignores the possibility of a severe inflation shock.

3 Income Withdrawal vs. Index-Linked Annuity

The income withdrawal option allows a pensioner to draw an income stream directly from the retirement fund instead of purchasing a whole life annuity. In the U.K. a pensioner who opts for income withdrawal can defer the purchase of a whole life annuity to age 75 at the latest.

The income withdrawal option is a means by which pensioners can avoid being locked into an asset offering an income linked to government bond yields at retirement, which many individuals may find unduly restrictive. Most pensioners opting for income withdrawal would choose to invest their fund in assets believed to offer higher returns (such as equities).

The risk premium, ρ , is defined as the difference between the expected real return on a diversified equity portfolio and the expected real yield on index-linked bonds. How large should the risk premium be? Wilkie (1994) considers this question and concludes that a figure of $\rho = 3$ percent per annum would be reasonable for the long-term risk premium on equities. Thus, if we assume a real yield of 3 percent in pricing index-linked annuities, it would be reasonable to assume an expected real return of 6 percent on our equity portfolio.

In this section we investigate the risks and benefits of drawing income from a fund invested entirely in equities by comparing the projected cash flows with those from an index-linked annuity. As in Section 2, we first adopt a deterministic approach using a continuous time model and then proceed to stochastic projections using a discrete time model.

3.1 Dividend Yield and Dividend Growth

An equity portfolio is expected to produce an increasing stream of dividend income. As equities are not redeemable, the expected return can be determined by evaluating the present value of the projected income stream over an infinite time horizon. We derive a formula for the real return from an equity portfolio in terms of the dividend yield and the rate of dividend growth, both of which are assumed to be constant. We first define the following terms for the equity portfolio:

- V = Current market value of the equity portfolio;
- d = Current rate of dividend income from the equity portfolio;
- y = d/V = Dividend yield on the equity portfolio;
- g = Real annual dividend growth; and
- w = Real annual return on the equity portfolio.

Clearly, we must have $w > g$.

The return on the portfolio, w , is the rate of interest at which the present value of the projected dividend income from the portfolio is equal to its market value, hence:

$$V = d \int_0^{\infty} \left(\frac{1+g}{1+w} \right)^{\tau} d\tau$$

which implies

$$1 = \gamma \int_0^{\infty} \left(\frac{1+g}{1+w} \right)^{\tau} d\tau$$

which further implies, after evaluating the integral, that

$$\ln(1+w) = \gamma + \ln(1+g). \quad (15)$$

Equation (15) implies that the interest earned on an equity portfolio can be split into two components: dividend yield and dividend growth. For the U.K. equity market as a whole, Thornton and Wilson (1992a) show that real dividend growth historically has averaged approximately 1 percent to 2 percent per annum. The balance between yield and growth, however, depends on the stocks selected; many equity funds have invested specifically to provide above average growth (hence lower yield) or above average yield (hence lower growth).

3.2 Deterministic Comparisons

3.2.1 Pensioners Who Live Off Dividends

A pensioner drawing income from an equity fund may wish to live off the dividends alone, to avoid selling stocks to meet income needs. For such a pensioner, a high yielding equity portfolio with zero real dividend growth¹³ may be preferable as the closest alternative to an index-linked annuity.

We consider an example where the real dividend growth is $g = 0$, the risk premium on the equity portfolio is $\rho = 3$ percent, $r = 3$ percent, and $w = 6$ percent. Thus, it follows from equation (15) that the income from the equity portfolio per unit of retirement fund is $\gamma = \ln(1.06) = 5.8$ percent.

The comparable income yield from an index-linked annuity is $1/\bar{a}_x^r$ which is shown in Table 5 for different retirement ages, using the male PA(90) life table. Table 5 shows that the dividend income from a high yielding equity portfolio is unlikely to match the income from an index-linked annuity at retirement ages above 50, the shortfall becoming greater as the age of retirement increases.

After retirement the real value of the fund will not change, given our assumption of a flat dividend yield and zero real dividend growth. It follows that when the pensioner later buys an annuity, the same fund (in

¹³An example of such a portfolio is the M&G Equity Income Fund, which in providing an above average income yield for its investors has achieved income growth roughly line with price inflation since its formation in 1972. M&G is one of the leading unit trusts (mutual funds) in the U.K.

Table 5
Index-Linked Annuity Income Yield $1/\bar{a}_x^r$

x	45	50	55	60	65	70	75
Yield	5.2%	5.8%	6.6%	7.6%	8.9%	10.8%	13.4%

real terms) will be available to purchase a cheaper annuity (because the pensioner is older). If the annuity is bought m years after retirement, the income (compared with buying an annuity at retirement) increases by the proportion:

$$\Delta_m = \frac{\bar{a}_x^r}{\bar{a}_{x+m}^r} - 1. \quad (16)$$

Table 6 shows the percentage increase in income obtained by deferring the purchase of an annuity, for selected values of x and m .

Table 6
Increase in Income for Pensioner Living Off Dividends

x	m					
	5	10	15	20	25	30
45	11.6%	26.8%	46.2%	72.4%	108.4%	158.6%
50	13.7%	31.1%	54.5%	86.8%	131.8%	
55	15.4%	36.0%	64.4%	103.9%		
60	17.9%	42.5%	76.8%			
65	20.9%	50.0%				
70	24.1%					

As expected from equation (16), a greater increase in income is achieved for a pensioner who retires later and defers the annuity purchase longer. The sacrifice of income before the annuity purchase, however, also increases with retirement age (Table 5). In the next section we use a method of comparison to determine when a higher overall income can be obtained by drawing income from an equity fund.

3.2.2 Pensioner Who Matches Annuity Income

We now assume the pensioner draws an income from the fund that matches the income from an index-linked annuity, selling assets (or reinvesting surplus dividends) as required. If the pensioner can obtain a

higher income after the purchase of an annuity, the income withdrawal option would be advantageous.

Using currency units that are linked to price inflation, we now define, for $t \geq 0$, the following:

f_t = Real value of the fund at time t , per unit of the initial fund at time 0.

We assume the pensioner retires at $t = 0$, hence $f_0 = 1$. If assets are continually sold to maintain the same income as from an index-linked annuity, f_t must satisfy the following differential equation:

$$-\frac{df_t}{dt} + \gamma f_t = \frac{1}{\bar{a}_x^r}. \quad (17)$$

Equation (17) yields the solution

$$f_t = e^{\gamma t} - \frac{\bar{s}_{t|}^{\gamma}}{\bar{a}_x^r} \quad (18)$$

where $\bar{s}_{t|}^{\gamma} = (e^{\gamma t} - 1)/\gamma$. The proportionate change in income on purchasing an annuity at age $x + m$ is given by:

$$\Delta_m = f_m \frac{\bar{a}_x^r}{\bar{a}_{x+m}^r} - 1. \quad (19)$$

Table 7 shows the increase in income on buying an annuity for different x and m . Table 7 suggests there is a critical retirement age above

Table 7
Increase in Income for a Pensioner
Who Matches Annuity Income

x	m					
	5	10	15	20	25	30
45	15.8%	38.1%	69.3%	115.4%	185.9%	297.1%
50	14.0%	32.2%	56.8%	91.0%	139.6%	
55	10.5%	22.6%	35.7%	47.7%		
60	6.0%	8.9%	3.3%			
65	-0.8%	-12.8%				
70	-11.5%					

which the pensioner suffers from taking the income withdrawal option. For pensioners who defer the purchase of the annuity until age 75, the projected increase in income is only 3 percent for a retirement age of 60 and falls to a reduction of 13 percent for a retirement age of 65. Thus, drawing income from an equity fund cannot be expected to provide a greater overall income for retirement ages much above 60 using the male PA(90) life table to price annuities.

It does not follow, however, that pensioners retiring at younger ages should always opt for income withdrawal because the investment risk involved may be unacceptable. In the next section we attempt to quantify this risk through simulation, using a stochastic model for the return obtained from the equity portfolio.

3.3 Stochastic Comparison

Deterministic projections, as used in Section 3.3, can be used to find when a fund invested in equities is likely to provide more income than a whole life annuity and the expected amount of this extra income. In this section we use stochastic projections to estimate:

- The probability that the pensioner is unable to match the income of an index-linked whole life annuity as a result of poor investment experience over the period in which income is drawn from the equity fund; and
- The amount of additional savings at retirement that would give reasonable assurance of maintaining an inflation-linked income in spite of poor investment experience.

3.3.1 Formulae for Projections

We require a stochastic model that will enable simulation of the market value of our equity portfolio and its dividend income. As in Section 2.4, we switch to a discrete time framework in which a stochastic approach is more easily accommodated.

For $t = 1, 2, \dots$, let

- V_t = Real market value of equity portfolio at time t ;
- d_t = Real dividend income from equity portfolio paid at time t ;
- y_t = d_t/V_t = Dividend yield on the equity portfolio at time t ;
- i_t = Real actuarial return on fund over $[t - 1, t)$;

c_t = Real growth in market value of assets over $[t - 1, t)$; and

g_t = Real annual dividend growth over $[t - 1, t)$.

As $d_t = d_{t-1}(1 + g_t)$ and $1 + c_t = V_t/V_{t-1}$, we get

$$c_t = (1 + g_t) \frac{y_{t-1}}{y_t} - 1. \quad (20)$$

Assuming that the pensioner withdraws or reinvests assets as needed to match the income from an index-linked annuity and that the sale or purchase of assets does not alter the composition of the portfolio (i.e., the relative weighting given to each individual stock does not change), leads to the following recurrence formula for f_t :

$$f_{t+1} = \left((1 + y_t) f_t - \frac{1}{\bar{a}_x^r} \right) (1 + c_{t+1}). \quad (21)$$

Following Thornton and Wilson (1992b) we define the real actuarial return on the fund as:

$$i_t = (1 + y_t)(1 + g_t) - 1. \quad (22)$$

Using equations (20) and (22), equation (21) can be rewritten as:

$$f_{t+1} = \left((1 + y_t) f_t - \frac{1}{\bar{a}_x^r} \right) \frac{y_t(1 + i_{t+1})}{y_{t+1}(1 + y_t)}. \quad (23)$$

3.3.2 Stochastic Model

Our stochastic model for the equity portfolio consists of the following two components:

- The factors $1 + i_t$ form a sequence of independent, identically distributed log-normal random variables, i.e., for $t = 1, 2, \dots$:

$$\ln(1 + i_t) \sim N(\mu_i, \sigma_i^2); \quad \text{and}$$

- The sequence $\ln(y_t)$ follows a first order autoregressive process with a normal residual for $t = 0, 1, 2, \dots$, as follows:

$$\ln(y_{t+1}) = \mu_y + k_y (\ln(y_t) - \mu_y) + \sigma_y Z_t \sqrt{1 - k_y^2} \quad (24)$$

where μ_y , σ_y and k_y are parametric constants representing, respectively, the mean of $\ln(y_t)$, its standard deviation, and the correlation between the $\ln(y_t)$'s in adjacent years.

The first component of the model focuses on the actuarial return rather than the return on market value, as historic data for the U.K. equity market show that actuarial returns have been much less variable. Thus, a model based on actuarial returns is likely to be a better description of the behavior of the U.K. equity market. The second component of the model is similar to the approach used by Wilkie in assuming that the dividend yield on U.K. equities tends to revert to a long-term average. This implies that the equity market tends to correct itself when stock prices are overvalued or undervalued relative to some par dividend yield, an assumption well supported by historic data for the U.K. equity market.

As in Section 3.3, we assume the pensioner invests in a portfolio of high yielding stocks from which the expected real dividend growth is zero. If dividends are payable annually in advance, the real return on the equity portfolio, w , satisfies the equation:

$$V_0 = d_0 \sum_{t=0}^{\infty} (1 + w)^{-t} = d_0 \frac{(1 + w)}{w},$$

which implies that $y_0 = w/(1 + w)$. We also must assume that the long-term par dividend yield is consistent with w , i.e.,

$$\mu_y = \ln(y_0) = \ln[w/(1 + w)]. \quad (25)$$

Equation (22) implies that when real dividend growth is zero, the actuarial return over any year is equal to the dividend yield at the start of the year. We therefore use the following estimate for μ_i :

$$\mu_i = \ln(1 + y_0) = \ln[1 + w/(1 + w)]. \quad (26)$$

Values for the other parameters used in our stochastic model are estimated from representative U.K. equity indices from 1919 to 1995¹⁴ as follows:

$$\sigma_i = 0.0675, \quad \sigma_d = 0.24, \quad \text{and } k_d = 0.50$$

which, respectively, are the standard deviation of the logarithm of the actuarial return, the standard deviation of the logarithm of the dividend yield at each year-end, and the correlation between the logarithm of dividend yields in adjacent years.

¹⁴The index used is the BZW equity index, a representative U.K. stock price index compiled by the investment bank, Barclays de Zoete Wedd. The index provides data on U.K. stock prices and yields from 1919 to the present.

3.3.3 Present Value of Surplus

We use the model described in Section 3.4.2 above to simulate values for the fund, f_t , at different durations from retirement. For each projection, we are interested in comparing the market value of the fund at any chosen duration with the money required to maintain an unchanged income after the purchase of an index-linked annuity.

In carrying out these simulations there are two complications:

- The pricing basis for index-linked annuities may change over time; and
- The pensioner can time the purchase of the annuity to exploit favorable changes in investment yields.

Strictly, we also require a stochastic model for the real yield on index-linked bonds to allow for random fluctuations in the pricing basis. Given that index-linked yields have been more stable than equity dividend yields and that most of the uncertainty is believed to arise from variability in equity returns, however, it may be adequate for our purpose to assume that the pricing basis does not alter between retirement and the purchase of the annuity.

We also ignore the second complication, for it assumes that pensioners can judge when equities are overpriced relative to index-linked bonds, something that even experienced fund managers may find difficult. As in Section 3.2, we perform projections assuming that the pensioner defers the purchase of an annuity for a fixed period.

On purchasing an annuity m years after retirement the surplus assets, u_m , are given by:

$$u_m = f_m - \frac{\ddot{a}_{x+m}^r}{\ddot{a}_x^r}. \quad (27)$$

If $u_m = 0$ the fund would be just sufficient to purchase an index-linked annuity providing the same real income. We assume that u_m can be negative as well as positive, which is implied from the use of our stochastic model.

Let $(PVU)_x(m)$ be the present value at retirement of the projected surplus or deficit using the same real interest used to price the index-linked annuities, i.e.,

$$(PVU)_x(m) = \frac{u_m}{(1+r)^m}. \quad (28)$$

3.3.4 Results of Simulations

Next we use simulations to estimate

$${}_m\xi_x(-h) = \Pr[(PVU)_x(m) < -h \mid \text{pensioner survives to age } x + m]$$

for $h = 0, 0.1, 0.2, 0.3$. Note, for example, that $\Pr[(PVU)_x(m) < 0]$ gives the probability that a pensioner who retires at age x will not be able to maintain the same real income after using the remaining fund to purchase an index-linked annuity m years after retirement.

One thousand simulations are performed for selected values of x and m using the stochastic model described in Section 3.4.2 and assuming: $r = 3$ percent and $w = 6$ percent. The initial dividend yield for each simulation and its long-term average value are as given in equation (25).

Table 8 is consistent with Table 6 in showing that the income withdrawal option becomes more risky as the age of retirement increases. The risk involved is significant even at a retirement age as low as 45. Tables 9, 10, and 11 show the estimated probabilities for $h = 0.1, 0.2$, and 0.3 , respectively.

Table 8
Simulation Estimates of ${}_m\xi_x(0)$

x	m					
	5	10	15	20	25	30
45	0.31	0.22	0.18	0.15	0.14	0.16
50	0.33	0.26	0.23	0.20	0.24	
55	0.40	0.33	0.34	0.36		
60	0.44	0.43	0.50			
65	0.49	0.59				

Using the same reasoning as Section 2.4.5, Tables 9, 10, and 11 give probabilities of not being able to maintain an inflation-linked income for a pensioner who had additional savings at retirement equal to h times the retirement fund and invested these savings in assets giving a guaranteed real return of 3 percent per annum.

Tables 8 through 11 show that each increment of 10 percent in the additional savings held at retirement significantly reduces the risk for any combination of x and m . For initial savings of 30 percent of the retirement fund, the risk is small for the younger retirement ages.

Table 9
Simulation Estimates of ${}_m\xi_x(-0.1)$

x	m					
	5	10	15	20	25	30
45	0.16	0.11	0.08	0.06	0.06	0.08
50	0.18	0.13	0.13	0.09	0.14	
55	0.23	0.17	0.19	0.23		
60	0.28	0.27	0.34			
65	0.32	0.41				

Table 10
Simulation Estimates of ${}_m\xi_x(-0.2)$

x	m					
	5	10	15	20	25	30
45	0.07	0.03	0.02	0.02	0.02	0.04
50	0.07	0.04	0.04	0.03	0.08	
55	0.08	0.07	0.08	0.13		
60	0.12	0.13	0.18			
65	0.17	0.24				

4 Summary and Conclusions

As expected, the three important factors that affect how a pensioner's retirement fund should be invested to provide an index-linked income stream for life are expected future price inflation, the pensioner's expected remaining life span, and the additional savings held by the pensioner. Apart from purchasing an index-linked annuity, we assume that the pensioner can either purchase a level whole life annuity or select income withdrawal.

4.1 Level Whole Life Annuities

The existence of price inflation has meant that the traditional whole-life annuity, providing a level monetary income, is no longer a risk-free option. Thus, we have adopted the newer index-linked annuity as the benchmark against which other options should be measured.

Table 11
Simulation Estimates of $m\xi_x(-0.3)$

x	m					
	5	10	15	20	25	30
45	0.02	0.01	0.00	0.00	0.01	0.01
50	0.02	0.01	0.01	0.01	0.03	
55	0.02	0.01	0.02	0.04		
60	0.03	0.04	0.08			
65	0.06	0.10				

Most U.K. pensioners still opt for a level annuity, however, and this is not an irrational preference. We show that the odds are in favor of obtaining a higher aggregate income from a level annuity, especially at older retirement ages. We also show that the existence of price inflation poses a significant longevity risk to the recipient of a level annuity, irrespective of whether the rate of inflation is constant or random. Thus, the purchase of a level annuity is perhaps only advisable for individuals with additional savings; we estimate that further assets of at least 20 percent of the retirement fund are necessary to insure against the risk of being unable to maintain an index-linked income stream until death.

4.2 Income Withdrawal

Our analysis of the income withdrawal option can be summarized as follows:

- The expected overall income from the equity fund is greater for retirement ages below a critical threshold at which the extra return from investing in equities is balanced by the extra cost of capital protection on death. Using the male PA(90) life table, this critical age is somewhere between 60 and 65;
- The expected extra income from the equity fund reduces as age of retirement increases, becoming more negative as the retirement age increases beyond the critical age;
- Even at retirement ages as young as 50, there remains a significant risk of underperforming relative to an index-linked annuity as a result of poor equity returns;

- Additional savings invested in assets providing a guaranteed real return can significantly reduce the risk of not being able to match the income from an index-linked annuity—savings of 30 percent of the retirement fund would reduce the risk to below 5 percent for retirement ages at which a higher overall income was expected.

Thus the greater the life expectancy at retirement, the greater the advantages of drawing income from an equity fund compared with a whole life annuity providing an income linked to bond yields. This accords with the actuarial viewpoint that equities are suitable assets for matching longer-term liabilities.

There is still a significant risk, however, that an equity fund may not be able to match the income from an index-linked annuity, even at young retirement ages. Again, the pensioner should have additional savings at retirement, with the minimum savings required varying between 20 percent and 30 percent of the retirement fund.

As a pensioner's actual life span is uncertain, the longevity risk associated with drawing income from a fund eventually will dominate other considerations. Thus, at some age the purchase of a whole life annuity becomes necessary. U.K. legislation does not allow the purchase of an annuity to be deferred beyond age 75; we have shown that a male pensioner may be unwise to defer the purchase beyond age 65.

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